

Some Notes on the Binomial Expansion

Expressions of the form $(a + b)^p$ often appear in physics problems, and often one of the terms a or b is much larger than the other one. We need a way to deal with this situation.

The exponent p can be a positive integer, but also it can be something else, like a negative integer, or a simple fraction, e.g. $\pm \frac{1}{2}$, $\pm \frac{1}{3}$, $\pm \frac{1}{4}$, corresponding to taking a root.

Without loss of generality (WLOG), we can assume that $a > b$. Then we can factor it out, so that the expression becomes $a^p (1 + x)^p$, where x is defined by $x = b/a$. [If a happened to be negative, we might want first to factor out the minus sign separately, so then we would be dealing with $|a| > 0$.] We can always perform this factorization, so that in physics problems, we can always arrange to be dealing with an expression of the form $(1 + x)^p$. Furthermore we have arranged it so that $|x| < 1$ and often $|x| \ll 1$ (x might be negative).

Recall from your calculus course the formula for the Taylor expansion of a function $f(x)$ about $x = 0$ (also called a McLaurin series):

$$f(x) = f(0) + f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \cdots + \frac{1}{n!} f^{(n)}(0)x^n + \cdots, \quad (1)$$

(the primes denote derivatives of successively higher order). When we calculate these derivatives for the expression $(1 + x)^p$ and substitute them into Eq. (1), we get

$$(1 + x)^p = 1 + px + \frac{1}{2!} p(p-1)x^2 + \frac{1}{3!} p(p-1)(p-2)x^3 + \frac{1}{4!} p(p-1)(p-2)(p-3)x^4 + \cdots. \quad (2)$$

The pattern of the higher order terms is clear. This formula is the binomial expansion.

In physics applications x is often small. Since powers of a small quantity are even smaller, it is often justified to terminate Eq. (2) after the second term. For example, in relativity when the speed v of an object is small compared to the speed of light c , so that

$\beta \equiv \frac{v}{c} \ll 1$, then we can approximate (take $p = \frac{1}{2}$, and $x = -\beta^2$)

$$\sqrt{1 - \beta^2} \approx 1 - \frac{1}{2} \beta^2. \quad (3)$$