

2006-08-30 Problem Solutions

Problem 1: Tipler 1-12

a) Since we know that the events occur at the same position in frame S' , let's use the "inverse" time transformation. For these two events that occur at times t_1' and t_2' in S' the corresponding times in S are

$$t_1 = \gamma \left(t_1' - \frac{v}{c^2} x_0' \right) \quad \text{and} \quad t_2 = \gamma \left(t_2' - \frac{v}{c^2} x_0' \right). \quad (1)$$

Subtract the first from the second to get the time interval in frame S . Since the x_0' term is in both, it cancels, and we get

$$t_2 - t_1 = \gamma (t_2' - t_1'). \quad (2)$$

This equation demonstrates what the problem asks. Since $\gamma > 1$, the time interval in S is longer than the time interval in S' .

b) The "inverse" transformation equations (1-21) are more convenient to use because the common position of the events in S' cancels when we subtracted the two equations above. If we had used the "forward" transformation equations (1-20) and subtracted we would have obtained a term $x_2 - x_1$, which is not zero.

Problem 2: Tipler 1-13

Coordinates of an event in S: $x = 75 \text{ m}$, $y = 18 \text{ m}$, $z = 4.0 \text{ m}$, $t = 2.0 \times 10^{-5} \text{ s}$.

Velocity of S' w.r.t. S: $v = 0.850c$, i.e. $\beta = 0.850$.

a) Find the coordinates of this event in S'.

The Lorentz transformation:

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

$$\text{Calculate } \gamma = \frac{1}{\sqrt{1 - 0.850^2}} = 1.898$$

Then coordinate transform is

$$x' = 1.898 \left[75.0 \text{ m} - (0.850c)(2.00 \times 10^{-5} \text{ s}) \right] = -9.537 \times 10^3 \text{ m};$$
$$y' = y = 18 \text{ m}; \quad z' = z = 4.00 \text{ m}.$$

The time transform is

$$t' = 1.898 \left[2.0 \times 10^{-5} \text{ s} - \frac{(0.85c)(75 \text{ m})}{c^2} \right] = 3.756 \times 10^{-5} \text{ s}.$$

b) Use the inverse transformation to go back to the frame S.

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right), \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

The answer book says $x = 75.8 \text{ m}$ and $t = 2.0 \times 10^{-5} \text{ s}$. The error in x is due to rounding, it says. Why is there no corresponding error in t ?

Problem 3:

Since we're given the space-time coordinates in the S' frame and asked for the coordinates in the S frame, we use the inverse Lorentz transformation

$$x = \gamma(x' + vt'), \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right), \quad y = y', \quad z = z'. \quad (3)$$

Evaluate

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{9}{25}\right)}} = \frac{1}{\sqrt{\frac{16}{25}}} = \frac{5}{4}. \quad (4)$$

So

$$\begin{aligned} x &= \frac{5}{4} \left(60 \text{ m} + \frac{3}{5} \cdot 3 \times 10^8 \frac{\text{m}}{\text{s}} \cdot 8 \times 10^{-8} \text{ s} \right) = 93 \text{ m}, \\ t &= \frac{5}{4} \left(8 \times 10^{-8} \text{ s} + \frac{3c}{5} \cdot \frac{60 \text{ m}}{c^2} \right) = \frac{5}{4} \left(8 \times 10^{-8} \text{ s} + \frac{3}{5} \cdot \frac{60 \text{ m}}{3 \times 10^8 \text{ m/s}} \right) = \frac{5}{4} (8 + 12) \times 10^{-8} \text{ s} \quad (5) \\ &= 2.5 \times 10^{-7} \text{ s}. \end{aligned}$$

Problem 4:

Start from Tipler's Eq. (1.10),

$$\gamma^2 (x - vt)^2 + y^2 + z^2 = c^2 \gamma^2 \left[t + \frac{(1 - \gamma^2)}{\gamma^2} \frac{1}{v} x \right]^2. \quad (6)$$

Expand the expressions in parenthesis & brackets.

$$\gamma^2 (x^2 - 2vxt + v^2 t^2) + y^2 + z^2 = c^2 \gamma^2 \left[t^2 + 2 \frac{(1 - \gamma^2)}{\gamma^2} \frac{1}{v} xt + \frac{(1 - \gamma^2)^2}{\gamma^4} \frac{1}{v^2} x^2 \right]. \quad (7)$$

Collect terms.

$$\begin{aligned} x^2 \left[\gamma^2 - c^2 \gamma^2 \frac{(1 - \gamma^2)^2}{\gamma^4} \frac{1}{v^2} \right] - 2xt \left[v\gamma^2 + c^2 \gamma^2 \frac{(1 - \gamma^2)}{\gamma^2} \frac{1}{v} \right] + y^2 + z^2 \\ = t^2 [-\gamma^2 v^2 + c^2 \gamma^2]. \end{aligned} \quad (8)$$

This equation is supposed to be the same as Tipler's Eq. (1-17). The coefficient of x^2 should be 1; the book derives the formula for γ from that on p. 23. Continue with the other coefficients. The coefficient of xt should be 0.

$$\begin{aligned}
0 &= v\gamma^2 + c^2\gamma^2 \frac{(1-\gamma^2)}{\gamma^2} \frac{1}{v} = v\gamma^2 + c^2(1-\gamma^2)\frac{1}{v}, \\
\gamma^2 \left(v - \frac{c^2}{v} \right) &= -\frac{c^2}{v}, \\
\gamma^2 &= \frac{-\frac{c^2}{v}}{v - \frac{c^2}{v}} = \frac{1}{1 - \frac{v^2}{c^2}} \quad \Rightarrow \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.
\end{aligned} \tag{9}$$

[Notice that on the last step of taking the square root, in principle there could have been a $\pm$ sign. We chose the positive root so that $\gamma \rightarrow 1$ when $v \rightarrow 0$.]

[In class we used the coefficient of t^2 to find γ .]