

2006-09-01 Problem Solutions

Problem 1: Tipler 1-22

(a) $L = L_p / \gamma$ (Equation 1-20)

$$\frac{L_p - L}{L_p} = \frac{L_p - (1/\gamma)L_p}{L_p} = 1 - \frac{1}{\gamma} = \frac{1}{2} \frac{v^2}{c^2} = \frac{1}{2} \frac{(3 \times 10^{-6})^2}{c^2} = 4.5 \times 10^{-12} = 4.5 \times 10^{-10} \%$$

(b) $\Delta t = \Delta t' / \gamma = \Delta t' \left[1 - \frac{1}{2} \frac{v^2}{c^2} \right] = 3.16 \times 10^7 s \left[1 - \frac{1}{2} (3 \times 10^{-6})^2 \right]$

$\approx 3.16 \times 10^7 s$ elapses on the pilot's clock also. The pilot's clock loses:

$$\begin{aligned} \Delta t' - \Delta t &= \Delta t' (1 - 1/\gamma) \approx \Delta t' \frac{1}{2} \frac{v^2}{c^2} \\ &= (3.16 \times 10^7 s)(1/2)(3 \times 10^{-6})^2 = 1.42 \times 10^{-4} s (1 \text{ min}/60 s) = 2.37 \times 10^{-6} \text{ min} \end{aligned}$$

Problem 2: Tipler 1-27

Using Equation 1-30, with L_{A_p} and L_{B_p} equal to the proper lengths of A and B and $L_A =$ length of A measured by B and $L_B =$ length of B measured by A.

$$L_A = L_{A_p} / \gamma = 100 \text{ m} / \sqrt{1 - (0.92c)^2} = 39.2$$

$$L_{B_p} = \gamma L_B = 36 / \sqrt{1 - (0.92c)^2 / c^2} = 91.9 \text{ m}$$

Problem 3: Tipler 1-46

a) The length contraction formula gives

$$L = L_p \sqrt{1 - \left(\frac{v}{c}\right)^2} = (100 \text{ m}) \sqrt{1 - (0.85)^2} = 52.7 \text{ m}. \quad (1)$$

b) If we put frame S' on the rocket ship going right, so that $v = 0.85c$, then the velocity of the rocket going left with respect to Earth is $u = -0.85c$. In the frame of the first ship, the velocity is

$$u' = \frac{u - v}{1 - \frac{v}{c^2} u} = \frac{-0.85c - 0.85c}{1 - \frac{(0.85c)(-0.85c)}{c^2}} = -0.987c. \quad (2)$$

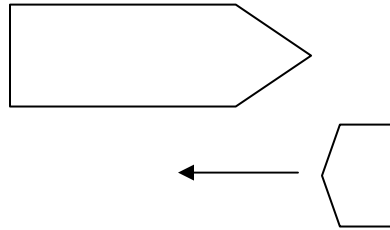
- c) Use the length contraction formula with the velocity found in Eq. (2).

$$L' = L_p \sqrt{1 - \left(\frac{u}{c}\right)^2} = (100 \text{ m}) \sqrt{1 - \left(\frac{1.70}{1.72}\right)^2} = 16.1 \text{ m}. \quad (3)$$

- d) In the Earth frame the two rockets have the same length, 52.7 m. Therefore their ends will be together after the time it takes for one ship to move its own contracted length.

$$\Delta t = \frac{L}{u} = \frac{52.7 \text{ m}}{0.85 \times 3 \times 10^8 \text{ m/s}} = 2.1 \times 10^{-7} \text{ s}. \quad (4)$$

- e)



In this frame, the longer ship is the one at rest and it is 100 m long. The shorter one is 16.1 m long [part(c)]. It is going left at speed $0.987c$ [part (b)].

Problem 4: Light going at angle θ in frame S

- a) In frame S , the components of the light velocity along the x and y axes are

$$u_x = c \cos \theta, \quad u_y = c \sin \theta.$$

Using the velocity transformation equations [Tipler Eqs. (1-25)], the components in frame S' are

$$u'_x = \frac{c \cos \theta - v}{1 - \frac{v}{c} \cos \theta}, \quad u'_y = \frac{1}{\gamma} \frac{c \sin \theta}{1 - \frac{v}{c} \cos \theta}. \quad (5)$$

The square of the magnitude of this velocity is the sum of the squares of these components. Introducing the notation $\beta = v/c$,

$$\begin{aligned}
u_x'^2 + u_y'^2 &= c^2 \left[\frac{(\cos \theta - \beta)^2}{(1 - \beta \cos \theta)^2} + \frac{\sin^2 \theta}{\gamma^2 (1 - \beta \cos \theta)^2} \right] \\
&= \frac{c^2}{(1 - \beta \cos \theta)^2} \left[\cos^2 \theta - 2\beta \cos \theta + \beta^2 + \sin^2 \theta (1 - \beta^2) \right] \\
&= \frac{c^2}{(1 - \beta \cos \theta)^2} \left[\underbrace{\cos^2 \theta + \sin^2 \theta}_1 - 2\beta \cos \theta + \beta^2 \underbrace{(1 - \sin^2 \theta)}_{\cos^2 \theta} \right] \\
&= \frac{c^2}{(1 - \beta \cos \theta)^2} \left[1 - 2\beta \cos \theta + \beta^2 \cos^2 \theta \right] \\
&= \frac{c^2}{(1 - \beta \cos \theta)^2} (1 - \beta \cos \theta)^2 = c^2.
\end{aligned}$$

Taking the square root gives the magnitude of the velocity of light in frame S' to be c, as expected.

b) Trig functions of the angle θ' can be found by taking ratios, e.g.
 $\tan \theta' = u_y' / u_x'$, $\cos \theta = u_x' / c$, $\sin \theta = u_y' / c$. I found the most useful one to be for the cosine.

$$\cos \theta' = \frac{u_x'}{c} = \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \quad \text{so} \quad \theta' = \text{Arccos} \left[\frac{\cos \theta - \beta}{1 - \beta \cos \theta} \right].$$

Evaluating this at $\theta = 1$, where $\cos \theta = 1$ gives

$$\theta' = \text{Arccos} \left[\frac{1 - \beta}{1 - \beta \cdot 1} \right] = \text{Arc cos}(1) = 0.$$

Evaluating it at $\theta = \pi/2$, where $\cos \theta = 0$ gives

$$\theta' = \text{Arccos} \left[\frac{0 - \beta}{1 - \beta \cdot 0} \right] = \text{Arc cos}(-\beta).$$

This angle is greater than $\pi/2$, i.e. it is in the 2nd quadrant. From this we can conclude that as θ increases, θ' increases more rapidly, so that θ' is greater than θ .

Here's are plots of θ' for $0 < \theta < \pi$ for different values of β .

red: $\beta = 0$ green: $\beta = 0.33$ yellow: $\beta = 0.66$ blue: $\beta = 0.99$

