

2006-09-08 Problem Solutions

Problem 1: Tipler 1-30

The formula for the Doppler effect for approach is [Equation (1-38)]

$$\frac{f}{f_0} = \sqrt{\frac{1+\beta}{1-\beta}}. \quad (1)$$

In this problem the “proper” wavelength (in the frame of the source) is $\lambda_0 = 650$ nm (red), and we want to find the speed of approach v that will shift it to $\lambda = 590$ nm (green). The ratio of these two wavelengths is $\lambda/\lambda_0 = 590/650$, so the corresponding frequency ratio is $f/f_0 = 650/590 = 1.102$ (because $f = c/\lambda$). We substitute into Eq. (1) and get

$$\begin{aligned} \sqrt{\frac{1+\beta}{1-\beta}} = 1.102 &\Rightarrow \frac{1+\beta}{1-\beta} = (1.102)^2 = 1.214 \Rightarrow 1+\beta = 1.214(1-\beta) \\ \Rightarrow 2.214\beta = 0.214 &\Rightarrow \beta = 0.0967 \Rightarrow v = 2.90 \times 10^7 \text{ m/s}. \end{aligned} \quad (2)$$

This is the speed at which to approach a red light so that it would appear green. It's 6.48×10^7 miles/hour, so you would get a ticket for speeding instead of running a red light.

Problem 2: Tipler 1-32

Since the frequency is increased, the star must be approaching the Earth.

We compare the formula in the problem, telling how much the frequency is increased, $f = 1.02f_0$, with the equation for the Doppler shift for an approaching object, Eq. (1) above and get

$$\sqrt{\frac{1+\beta}{1-\beta}} = 1.02. \quad (3)$$

We have to solve this for β . We could solve it by the method that was used in the previous problem: square the equation, clear is of fractions, rearrange it to solve for β and get an answer. But let's observe that the frequency changes by only 2%, so β must not be very big. Therefore here is an opportunity to use the binomial expansion!

$$\frac{1+\beta}{1-\beta} = (1+\beta)(1-\beta)^{-1} = (1+\beta)[1+(-1)(-\beta)+\dots] = (1+\beta)^2 = 1+2\beta+\dots \quad (4)$$

Notice that on the last step we didn't square out $(1+\beta)^2$ completely; we dropped the β^2 term. That's because in the binomial expansion step, we already dropped small terms of the order of β^2 , so to be consistent we have to drop the β^2 term here also. Next we take

the square root of Eq. (4), using the binomial expansion again, to get the quantity in Eq. (3).

$$\sqrt{\frac{1+\beta}{1-\beta}} = [1 + 2\beta + \dots]^{1/2} = 1 + \frac{1}{2}(2\beta) + \dots = 1 + \beta. \quad (5)$$

We compare Eqs. (3) and (5) and get

$$\beta = 0.02 \quad \text{or} \quad v = 0.02c = 6.00 \times 10^6 \text{ m/s}. \quad (6)$$

[Doing the calculation “exactly” without using the binomial expansion gives $\beta = 0.0198$.]

Problem 3: Tipler 1-40

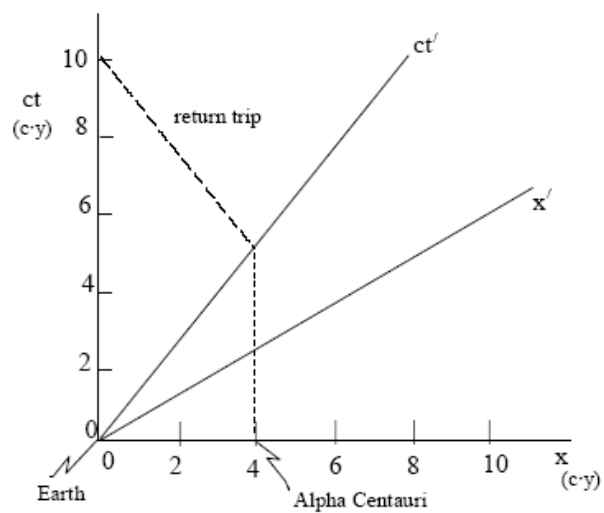
(a) Alpha Centauri is $4 \text{ c} \cdot \text{y}$ away, so the traveler went $L = \sqrt{1-\beta^2} (8 \text{ c} \cdot \text{y})$ in 6 y , or

$$\begin{aligned} 8 \text{ c} \cdot \text{y} \sqrt{1-v^2/c^2} &= v(6 \text{ y}) \\ \sqrt{1-v^2/c^2} &= v(6/8 \text{ c}) = (3/4)(v/c) \\ &= 1 - \beta^2 = (3/4)^2 \beta^2 \\ (3/4)^2 \beta^2 + \beta^2 &= 1 \\ \beta^2 &= 1/(1 + 0.5625) \\ v &= 0.8c \end{aligned}$$

(b) $\Delta t = \gamma \Delta t_0 = \gamma(6y)$ and $\gamma = 1/\sqrt{1-\beta^2} = 1.667$

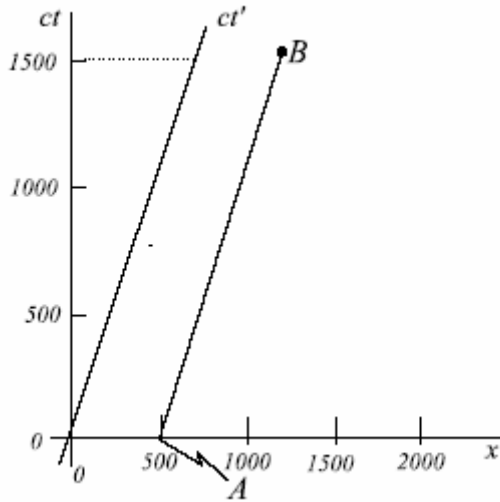
$\Delta t = 1.667(6y) = 10y$ or $4y$ older than the other traveler.

(c)



Problem 4: Tipler 1-45

Two events (explosions) in frame S: A :($x_1 = 480$ m, $ct_1 = 0$); B: ($x_2 = 1200$ m, $ct_2 = 1500$ m). Here is the spacetime graph with these two events plotted on it.



In frame S' , the events occur at the same point. We draw the straight line between the two events; that's a line of constant x' , but it isn't $x' = 0$ because it doesn't go through the origin. The slope of this line is

$$\text{slope}(AB) = \frac{1500 - 0}{1200 - 480} = 2.08.$$

The line $x' = 0$, which is the ct' axis, is parallel to the line AB, so has the same slope. General theory (p. 29) says the slope of the ct' axis is $1/\beta$, so

$$\beta = \frac{1}{2.08} = 0.480.$$

and then

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = 1.140.$$

To “calibrate” the ct' axis, we can do the following. Use the Lorentz transformation equation $t' = \gamma(t - vx/c^2)$, rewritten as $ct' = \gamma(ct - \beta x)$, we calculate the time of event A in S' ,

$$ct'_1 = 1.14(0 - 0.48 \cdot 480 \text{ m}) = -263 \text{ m},$$

and of event B

$$ct_2' = 1.14(1500 \text{ m} - 0.48 \cdot 1200 \text{ m}) = 1053 \text{ m} .$$

Therefore

$$c\Delta t' = 1053 \text{ m} - (-263 \text{ m}) = 1316 \text{ m}$$

and

$$\Delta t' = 4.39 \times 10^{-6} \text{ s} .$$

To check, since the events are at the same point in S' , $\Delta t'$ is a proper time interval and is related to Δt by the time dilation formula.

$$\Delta t' = \frac{\Delta t}{\gamma} = \frac{5 \times 10^{-6} \text{ s}}{1.14} = 4.39 \times 10^{-6} \text{ s} .$$

It checks.