

2006-09-18 Problem Solutions

Problem 1: Energy in terms of momentum

Here is one way to obtain this relation; there are other ways.

Relativistic formulas for momentum and energy of a particle with rest mass m are

$$\mathbf{p} = \gamma_u m \mathbf{u}, \quad E = \gamma_u mc^2. \quad (1)$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (2)$$

Square the two quantities in Eq. (1).

$$p^2 \equiv \mathbf{p} \cdot \mathbf{p} = \frac{m^2 u^2}{1 - \frac{u^2}{c^2}} = \frac{(m^2 c^2) \frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}}, \quad E^2 = \frac{(mc^2)^2}{1 - \frac{u^2}{c^2}}. \quad (3)$$

Solve the 2nd of Eqs. (3) for the ratio u^2/c^2 .

$$\frac{E^2}{(mc^2)^2} = \frac{1}{1 - \frac{u^2}{c^2}}, \quad 1 - \frac{u^2}{c^2} = \frac{(mc^2)^2}{E^2}, \quad \frac{u^2}{c^2} = 1 - \frac{(mc^2)^2}{E^2}. \quad (4)$$

Substitute the formulas from Eq. (4) into the 1st of Eqs. (3) for p^2 .

$$p^2 = m^2 c^2 \frac{1 - \frac{(mc^2)^2}{E^2}}{\left(\frac{(mc^2)^2}{E^2} \right)} = m^2 c^2 \left[\frac{E^2}{(mc^2)^2} - 1 \right] \quad (5)$$

$$p^2 c^2 = (mc^2)^2 \left[\frac{E^2}{(mc^2)^2} - 1 \right] = E^2 - (mc^2)^2 \quad (6)$$

Rearranging gives the desired result,

$$E^2 = (mc^2)^2 + (pc)^2. \quad (7)$$

Problem 2: Tipler 2-8

The work done to accelerate a particle from $u = 0$ to some final speed u equals the kinetic energy at speed u : $W_{0 \rightarrow u} = E_k(u)$. Therefore the work done to accelerate a particle from some initial speed u_0 to a final speed u is the difference of the kinetic energy values at the final speed and the initial speed:

$$W_{u_0 \rightarrow u} = E_k(u) - E_k(u_0). \quad (8)$$

Using the equation before Eq. (2-9) in the book for the kinetic energy, we can write

$$W_{u_0 \rightarrow u} = mc^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - \frac{1}{\sqrt{1 - \frac{u_0^2}{c^2}}} \right]. \quad (9)$$

For a proton, $mc^2 = 938 \text{ MeV}$.

a) For $u_0 = 0.15c$ to $u = 0.16c$

$$\begin{aligned} W_{u_0 \rightarrow u} &= (938 \text{ MeV}) \left[\frac{1}{\sqrt{1 - (0.16)^2}} - \frac{1}{\sqrt{1 - (0.15)^2}} \right] \\ &= (938 \text{ MeV}) \times (1.61 \times 10^{-3}) = 1.51 \text{ MeV}. \end{aligned} \quad (10)$$

b) For $u_0 = 0.85c$ to $u = 0.86c$,

$$\begin{aligned} W_{u_0 \rightarrow u} &= (938 \text{ MeV}) \left[\frac{1}{\sqrt{1 - (0.86)^2}} - \frac{1}{\sqrt{1 - (0.85)^2}} \right] \\ &= (938 \text{ MeV}) \times (6.13 \times 10^{-2}) = 57.5 \text{ MeV} \end{aligned} \quad (11)$$

c) For $u_0 = 0.95c$ to $u = 0.96c$,

$$\begin{aligned}
 W_{u_0 \rightarrow u} &= (938 \text{ MeV}) \left[\frac{1}{\sqrt{1 - (0.96)^2}} - \frac{1}{\sqrt{1 - (0.95)^2}} \right] \\
 &= (938 \text{ MeV}) \times (3.69 \times 10^{-1}) = 346 \text{ MeV}
 \end{aligned} \tag{12}$$

The point of this problem is to demonstrate that it takes a lot more work at higher energies to achieve the same change in the speed. This is because of the vertical asymptote in the energy vs speed graph at $u = c$.

Problem 3: Tipler 2-9

(a) The problem fails to tell us the mass of a gold nucleus, but to an excellent approximation, most nuclei have a mass of about 1 u per nucleon. Thus the mass of ^{197}Au would be (see p. 89) about $197 \text{ u} = 197 \times 0.9315 \text{ GeV}/c^2 = 183.5 \text{ GeV}/c^2$.

The energy is given in the problem to be 200 GeV/nucleon , so $E = 197 \times 200 \text{ GeV} = 39,400 \text{ GeV}$. Therefore the factor γ is given by

$$\gamma = \frac{E}{mc^2} = \frac{39,400 \text{ GeV}}{183.5 \text{ GeV}} = 214.7$$

We then find the velocity u/c which is about

$$\frac{u}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{214.7^2}} = 0.9999892$$

So these particles are extremely relativistic.

(b) The momentum is most quickly computed using

$$\begin{aligned}
 pc &= \sqrt{E^2 - (mc^2)^2} = \sqrt{(39,400 \text{ GeV})^2 - (183.5 \text{ GeV})^2} = 39,400 \text{ GeV} \\
 p &= 39,400 \frac{\text{GeV}}{c}
 \end{aligned}$$

The momentum is to the right for one nucleus, and to the left for the other nucleus.

One sees here another characteristic of extremely relativistic particles: the rest mass energy term is negligible compared to the total energy term. To an accuracy of 4 significant figures, the 2nd term under the square root can be neglected.

(c) Let's work out the momentum of the second nucleus as viewed by in the frame where the first nucleus is at rest. Since the second nucleus is moving to the left, it has a momentum of $p_x = -39,400 \text{ GeV}/c$. We now simply use the Lorentz transformation to

derive the momentum and energy in a frame moving with velocity 0.9999892 of the speed of light. The factor γ has already been worked out.

$$E' = \gamma(E - vp) = 214.7(39,400 \text{ GeV} - 0.9999892(-39,400 \text{ GeV})) = 1.692 \times 10^7 \text{ GeV}$$

$$\begin{aligned} p' &= \gamma(p - vE/c^2) = 214.7(-39,400 \text{ GeV}/c - 0.9999892(39,400 \text{ GeV}/c)) \\ &= -1.692 \times 10^7 \text{ GeV}/c \end{aligned}$$

The momentum of this left going particle is more than 400 times larger in frame S' than in the original frame S . Again, as in part (b), for this extremely relativistic particle, E and pc are essentially equal; the rest mass term is negligible.

Problem 4: Tipler 2-13

The total energy is $E = \gamma mc^2$, and the rest mass energy is mc^2 . For this particle the total energy is twice the rest mass energy, so $\gamma = 2$.

a)

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} = 2; & \frac{1}{1 - \left(\frac{u}{c}\right)^2} &= 4; & 1 - \left(\frac{u}{c}\right)^2 &= \frac{1}{4}; \\ \left(\frac{u}{c}\right)^2 &= \frac{3}{4}; & \frac{u}{c} &= \frac{\sqrt{3}}{2} = 0.866. \end{aligned} \tag{13}$$

$$\text{b) Momentum is } p = \gamma mu = \gamma mc \frac{u}{c} = 2mc \frac{\sqrt{3}}{2} = \sqrt{3}mc. \tag{14}$$