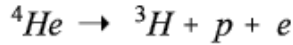


2006-09-20 Problem Solutions

Problem 1: Tipler 2-16



$$\begin{aligned} Q &= [m({}^3\text{H}) + m_p + m_e - m({}^4\text{He})]c^2 \\ &= 2809.450\text{ MeV} + 938.280\text{ MeV} + 0.511\text{ MeV} - 3728.424\text{ MeV} = 19.817\text{ MeV} \end{aligned}$$

Problem 2: Tipler 2-19

$$\begin{aligned} \text{(a)} \quad \Delta m &= m({}^4\text{He}) - 2m({}^2\text{H}) = \frac{m({}^4\text{He})c^2 - 2m({}^2\text{H})c^2}{uc^2} u \\ &= [3727.409\text{ MeV} - 2 \times 1875.628\text{ MeV}]u / 931.5\text{ MeV} \\ &= -0.0256 u \end{aligned}$$

$$\text{(b)} \quad \Delta E = |\Delta m|c^2 = (0.0256 uc^2)(931.5\text{ MeV}/uc^2) = 23.8\text{ MeV}$$

$$\text{(c)} \quad \frac{dN}{dt} = \frac{P}{\Delta E} = \frac{1\text{ W}}{23.847\text{ MeV}} \times \frac{1\text{ eV}}{1.602 \times 10^{-19}\text{ J}} = 2.62 \times 10^{11}/s$$

Problem 3: Tipler 2-21

Here is the solution from the author's solution manual. It uses conservation laws in the lab frame rather than using the invariant rest mass and the zero momentum frame.

How is the threshold condition expressed in this frame?

Conservation of energy requires that $E_i^2 = E_f^2$, or

$(p_i c)^2 + (2m_p c^2)^2 = (p_f c)^2 + (2m_p c^2 + m_\pi c^2)^2$ and conservation of momentum requires that

$p_i = p_f$, so

$$4(m_p c^2)^2 = 4(m_p c^2)^2 + 2m_p c^2 \times 2m_\pi c^2 + (m_\pi c^2)^2$$

$$0 = 2m_p c^2 \times 2m_\pi c^2 + (m_\pi c^2)^2$$

$$0 = m_\pi c^2 \left(2 + \frac{m_\pi c^2}{2m_p c^2} \right) = m_\pi c^2 \left(2 + \frac{m_\pi}{2m_p} \right)$$

Thus, $m_\pi c^2 (2 + m_\pi/2m_p)$ is the minimum or threshold energy E_t that a beam proton must have to produce a π^0 .

$$E_t = m_\pi c^2 \left(2 + \frac{m_\pi c^2}{2m_p c^2} \right) = 135 \text{ MeV} \left(2 + \frac{135}{2(938)} \right) = 280 \text{ MeV}$$

Here's the solution using the invariant mass and the zero-momentum frame, as discussed in class using the example of the Berkeley Bevatron.

In the center of momentum frame, before collision, two protons approach each other with equal and opposite velocities. After collision, at threshold, there are three particles at rest: two protons and one π^0 . The energy-momentum invariant is

$$\text{invariant} = E_{\text{total}}^2 - \underbrace{p_{\text{total}}^2}_0 c^2 = \left[(2m_p + m_\pi) c^2 \right]^2. \quad (1)$$

In the lab frame before the collision, proton 1, with both rest mass energy and kinetic energy, is incident on proton 2 at rest. The energy-momentum invariant, evaluated in this frame is

$$(E_1 + m_p c^2)^2 - c^2 p_1^2 = \text{invariant}. \quad (2)$$

But for proton 1,

$$E_1^2 = (m_p c^2)^2 + p_1^2 c^2 \quad \text{or} \quad p_1^2 c^2 = E_1^2 - (m_p c^2)^2. \quad (3)$$

Substitute Eq. (3) in Eq. (2), expand it, equate it to the value found in Eq.(1), and solve for the threshold total energy of the incident proton in the lab frame.

$$\begin{aligned} E_1^2 + 2(m_p c^2)E_1 + (m_p c^2)^2 - E_1^2 + (m_p c^2)^2 &= \left[(2m_p + m_\pi) c^2 \right]^2, \\ 2(m_p c^2)E_1 + 2(m_p c^2)^2 &= 4(m_p c^2)^2 + 4(m_p c^2)(m_\pi c^2) + (m_\pi c^2)^2, \\ 2(m_p c^2)E_1 &= 2(m_p c^2)^2 + 4(m_p c^2)(m_\pi c^2) + (m_\pi c^2)^2, \\ E_1 &= m_p c^2 + 2(m_\pi c^2) + \frac{(m_\pi c^2)^2}{2m_p c^2}. \end{aligned} \quad (4)$$

The first term in the last equation is the proton rest mass energy, so the rest of the equation is the threshold kinetic energy to get this reaction to occur.

$$E_k = 2(m_\pi c^2) + \frac{(m_\pi c^2)^2}{2m_p c^2} = 2(135 \text{ MeV}) + \frac{(135 \text{ MeV})^2}{2 \cdot 938 \text{ MeV}} = 279.7 \text{ MeV}. \quad (5)$$

Problem 4: Tipler 2-46

The system looks like Fig. 2-10 (p. 94), except without the specific values of m and u used in that figure.

a) In inertial frame S there are two particles, each of mass m and moving in opposite directions with velocities $+u$ and $-u$.

Choose inertial frame S' to have velocity $v = -u$, so that the particle on the right is at rest in frame S' . Then the velocity of the particle not at rest (the one on the left) is obtained from the velocity transformation to be

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} = \frac{u - (-u)}{1 - \frac{u(-u)}{c^2}} = \frac{2u}{1 + \frac{u^2}{c^2}}. \quad (6)$$

Then

$$u'^2 = \frac{4u^2}{\left(1 + \frac{u^2}{c^2}\right)^2}, \quad (7)$$

so

$$1 - \frac{u'^2}{c^2} = 1 - \frac{4u^2}{\left(1 + \frac{u^2}{c^2}\right)^2} = \frac{\left(1 + \frac{u^2}{c^2}\right)^2 - 4\frac{u^2}{c^2}}{\left(1 + \frac{u^2}{c^2}\right)^2} = \frac{\left(1 - \frac{u^2}{c^2}\right)^2}{\left(1 + \frac{u^2}{c^2}\right)^2} \quad (8)$$

and

$$\sqrt{1 - \frac{u'^2}{c^2}} = \frac{\left(1 - \frac{u^2}{c^2}\right)}{\left(1 + \frac{u^2}{c^2}\right)}. \quad (9)$$

b) In S the total energy and total momentum before the collision are

$$E_{\text{total}} = 2 \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad p_{\text{total}} = 0. \quad (10)$$

In S' , using the Lorentz transform for momentum, $p' = \gamma \left(p - \frac{v}{c^2} E \right)$, the momentum before the collision is

$$p'_{\text{before}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \left(0 - \frac{(-u)}{c^2} \frac{2mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = \frac{2mu}{1 - \frac{u^2}{c^2}}. \quad (11)$$

c) After the inelastic collision, there is only one object, with some rest mass M . Since the total momentum was zero before the collision in S , after the collision M is at rest. In S' , it has velocity $+u$ (apply the velocity transformation to an object that is at rest in S). In S' its momentum is

$$p'_{\text{after}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} Mu. \quad (12)$$

By conservation of momentum in S' , Eqs. (11) and (12) must be equal, so the rest mass of the combined mass after the collision is

$$M = \frac{2m}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (13)$$

d) In S the energy before the collision is given in Eq. (10). After the collision there is one mass M at rest, so

$$E_{\text{after}} = Mc^2 = \frac{2mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}. \quad (14)$$

Therefore, in S , $E_{\text{before}} = E_{\text{after}}$.

In S' before the collision the mass m on the right is at rest and the one on the left has the velocity given in Eq. (6), so, using Eq. (9), the total energy is

$$E'_{\text{before}} = mc^2 + \frac{mc^2}{\sqrt{1 - \frac{u'^2}{c^2}}} = mc^2 \left[1 + \frac{1 + \frac{u^2}{c^2}}{1 - \frac{u^2}{c^2}} \right] = mc^2 \frac{2}{1 - \frac{u^2}{c^2}}. \quad (15)$$

After the collision, in S' there is one mass M moving at speed u , so, using Eq. (13), its energy is

$$E'_{\text{after}} = \frac{Mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{2m}{\sqrt{1 - \frac{u^2}{c^2}}} c^2 = mc^2 \frac{2}{1 - \frac{u^2}{c^2}}. \quad (16)$$

Therefore, also in S' , $E'_{\text{before}} = E'_{\text{after}}$.