

2006-09-26 Problem Solutions

Problem 1: Tipler 2-37

The speed v of the satellite is:

$$v = 2\pi R/T = 2\pi(6.37 \times 10^6 \text{ m})/(90 \text{ min} \times 60 \text{ s/min}) = 7.42 \times 10^3 \text{ m/s}$$

Special relativistic effect:

After one year the clock in orbit has recorded time $\Delta t' = \Delta t/\gamma$, and the clocks differ by:

$$\Delta t - \Delta t' = \Delta t - \Delta t/\gamma = \Delta t(1 - 1/\gamma) \approx \Delta t(v^2/2c^2), \text{ because } v \ll c. \text{ Thus,}$$

$$\Delta t - \Delta t' = (3.16 \times 10^7 \text{ s})(7.412 \times 10^3)^2 / (2)(3.0 \times 10^8 \text{ m/s})^2 = 0.00965 \text{ s} = 9.65 \text{ ms}$$

Due to special relativity time dilation the orbiting clock is behind the Earth clock by 9.65 ms.

General relativistic effect:

$$\frac{\Delta f}{f_0} = \frac{gh}{c^2} = \frac{(9.8 \text{ m/s}^2)(3.0 \times 10^5 \text{ m})}{(3.0 \times 10^8 \text{ m/s})^2} = 3.27 \times 10^{-11} \text{ s/s}$$

In one year the orbiting clock gains

$$(3.27 \times 10^{-11} \text{ s/s})(3.16 \times 10^7 \text{ s/y}) = 1.03 \text{ ms}$$

The net difference due to both effects is a slowing of the orbiting clock by $9.65 - 1.03 = 8.62 \text{ ms}$.

Problem 2: Tipler 2-48

(a) $p_i = 0 = E/c - Mv$ or $v = E/Mc$

(b) The box moves a distance $\Delta x = v\Delta t$, where $\Delta t = L/c$, so

$$\Delta x = (E/Mc)(L/c) = EL/Mc^2$$

One needs to be careful about signs here. If the radiation is emitted from the left end [see part (c)] and travels right, then the box displaces left by the amount Δx .

- (c) Let the center of the box be at $x = 0$. Radiation of mass m is emitted from the left end

of the box (e.g.) and the center of mass is at: $x_{CM} = \frac{M(0) + m(L/2)}{M + m} = \frac{mL}{2(M + m)}$

When the radiation is absorbed at the other end the center of mass is at:

$$x_{CM} = \frac{M(EL/Mc^2) + m(L/2 - EL/Mc^2)}{M + m}$$

Equating the two values of x_{CM} (if CM is not to move) yields:

$$m = (E/c^2)(1 - E/Mc^2)$$

Because $E \ll Mc^2$, then $m \approx E/c^2$ and the radiation has this mass.

I think there should be a minus sign here. If the center of the box is at 0, the left end is at $-L/2$. This minus sign carries through to subsequent equations.

Problem 3: Tipler 2-53

Here is the solution from the authors' solution manual. It uses the invariant mass and the conservation laws.

- (a) Energy and momentum are conserved.

Initial system: $E = Mc^2$, $p = 0$

invariant mass: $(Mc^2)^2 = E^2 - (pc)^2 = (Mc^2)^2 + 0$

Final system:

invariant mass: $(2mc^2)^2 = (Mc^2)^2 + 0$

For 1 particle (from symmetry)

$$(mc^2)^2 = (Mc^2/2)^2 - p^2c^2 = (Mc^2/2)^2 - (\gamma u c)^2$$

Rearranging,

$$1 = \left(\frac{Mc^2}{2mc^2} \right)^2 - (\gamma u/c)^2 \Rightarrow \gamma^2 = \left(\frac{Mc^2}{2mc^2} \right)^2 = \frac{1}{1 - u^2/c^2}$$

Solving for u ,

$$u = \left[1 - \left(\frac{2mc^2}{Mc^2} \right)^2 \right]^{1/2} c$$

They didn't use this. And it's wrong. The invariant mass is not the sum of the rest mass energies when the particles are moving.

An m is missing here.

"Symmetry" means after the decay each particle gets half of the initial energy. The total momentum must be zero, so the two particles have equal and opposite momenta and velocities, and equal energies.

- (b) Energy and momentum are conserved.

Initial system: $E = 4mc^2$

invariant mass: $(Mc^2)^2 = (4mc^2)^2 - (pc)^2$

Final system:

invariant mass: $(2mc^2)^2 = (4mc^2)^2 - (pc)^2$

where $(pc)^2 = (4mc^2)^2 - (Mc^2)^2$

$$\frac{u}{c} = \frac{pc}{E} = \frac{[(4mc^2)^2 - (Mc^2)^2]^{1/2}}{4mc^2}$$

$$\left(\frac{u}{c}\right)^2 = \frac{(4mc^2)^2 - (Mc^2)^2}{(4mc^2)^2} = 1 - \left(\frac{Mc^2}{4mc^2}\right)^2$$

$$u = \left[1 - \left(\frac{Mc^2}{4mc^2}\right)^2\right]^{1/2} c$$

I think the authors' solution to (b) is complete garbage. It ignores their own comment at the start of Example 2-12: "Remember that the rest energy of a system of particles is not the sum of the rest energies of the individual particles, if they move relative to one another". Also, there are two particles after the decay, so there should be two answers.

Alternate solution to (a):

This can be solved using the energy and momentum conservation laws. The total momentum is zero before the decay; therefore the two identical particles emitted from the decay must have momenta and velocities of equal magnitude and opposite directions.

Call their common speed u . Therefore the total energy after the decay is, $E = \gamma_u mc^2$.

Equate this to the total energy before the decay and solve for u .

$$2\gamma_u mc^2 = Mc^2 \quad \Rightarrow \quad \gamma_u \equiv \frac{1}{\sqrt{1-u^2/c^2}} = \frac{M}{2m}$$

$$\Rightarrow \quad \sqrt{1-u^2/c^2} = \frac{2m}{M} \quad \Rightarrow \quad 1 - \frac{u^2}{c^2} = \left(\frac{2m}{M}\right)^2$$

$$\Rightarrow \quad \frac{u}{c} = \sqrt{1 - \left(\frac{2m}{M}\right)^2}.$$

Alternate solution to (b).

Before the decay there is one particle with total initial energy (rest mass plus kinetic)

$E_i = 4mc^2$. Find the velocity of this particle.

$$\gamma_u Mc^2 = 4mc^2 \quad \Rightarrow \quad \frac{1}{\gamma_u} = \frac{M}{4m} \quad \Rightarrow \quad 1 - \frac{u^2}{c^2} = \left(\frac{M}{4m}\right)^2$$

$$\Rightarrow \quad u = \sqrt{1 - \left(\frac{M}{4m}\right)^2} c.$$

Now, transform into a frame S' moving with velocity $v = u$, where the particle M is at rest. In S' when M decays, the two decay products move with equal and opposite velocities. We already calculated their velocities in part (a). So in frame S'

$$u_1' = \sqrt{1 - \left(\frac{2m}{M}\right)^2} c, \quad u_2' = -\sqrt{1 - \left(\frac{2m}{M}\right)^2} c.$$

Now we transform these velocities back to the lab frame, using the inverse velocity transformation $[u_j = (u_j' + v) / (1 + u_j' v / c^2)]$, where $j = 1, 2$ for the two particles.. v is the velocity u calculated above.

$$u_1 = \frac{u_1' + v}{1 + \frac{u_1' v}{c^2}} = \frac{\sqrt{1 - \left(\frac{2m}{M}\right)^2} c + \sqrt{1 - \left(\frac{M}{4m}\right)^2} c}{1 + \sqrt{1 - \left(\frac{2m}{M}\right)^2} \sqrt{1 - \left(\frac{M}{4m}\right)^2}} = \frac{\sqrt{1 - \left(\frac{2m}{M}\right)^2} + \sqrt{1 - \left(\frac{M}{4m}\right)^2}}{1 + \sqrt{1 - \left(\frac{2m}{M}\right)^2} \sqrt{1 - \left(\frac{M}{4m}\right)^2}} c$$

$$u_2 = \frac{u_2' + v}{1 + \frac{u_2' v}{c^2}} = \frac{-\sqrt{1 - \left(\frac{2m}{M}\right)^2} c + \sqrt{1 - \left(\frac{M}{4m}\right)^2} c}{1 - \sqrt{1 - \left(\frac{2m}{M}\right)^2} \sqrt{1 - \left(\frac{M}{4m}\right)^2}} = \frac{-\sqrt{1 - \left(\frac{2m}{M}\right)^2} + \sqrt{1 - \left(\frac{M}{4m}\right)^2}}{1 - \sqrt{1 - \left(\frac{2m}{M}\right)^2} \sqrt{1 - \left(\frac{M}{4m}\right)^2}} c$$

These answers aren't very pretty (but at least they have the correct units). Since the two velocities are different, the particles do move relative to each other and their invariant energy is not the sum of their rest energies.

One other way to work this problem is to find the momentum and energy of each particle in frame S' , find the γ factor for the relative motion of the two frames, then use the inverse transformation equations for momentum and energy to get the momentum and energy of each particle in frame S , and as the last step use the equation $u/c = pc/E$ to get the velocities in frame S . I did that and got the same result as above. When I have time, I'll write up the details.

Still another way to work the problem is to use only frame S and write the momentum and energy conservation equations in that frame. These conservation equations can be written either in terms of velocity or momentum. One would then solve them for the final velocities. The conservation equations are not difficult to write but solving for the final velocities seems to be difficult, so I haven't finished it yet. I think the lesson is that that one should transform into the frame where some things are simpler (such as the frame where the total momentum is zero), solve the problem there and then transform back to the original frame.