

2006-09-02 Problem Solutions

Problem 1: Tipler 3-15

a) According to the “Wien displacement law”, Eq. (3-20) in the text, the wavelength λ_m at the maximum of the spectral distribution and the temperature T are related by

$$\lambda_m T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}. \quad (1)$$

For $T = 2.7 \text{ K}$, $\lambda_m = \frac{2.898 \times 10^{-3} \text{ m} \cdot \text{K}}{2.7 \text{ K}} = 1.073 \times 10^{-3} \text{ m} = 107.3 \text{ mm}$.

b) The frequency is $f = \frac{c}{\lambda} = \frac{2.998 \times 10^8 \text{ m/s}}{1.073 \times 10^{-3} \text{ m}} = 2.794 \times 10^{11} \text{ Hz} = 0.2794 \text{ THz}$.

c) The Stefan-Boltzmann law, Eq. (3-19) of the text, gives the total power per unit area in this background radiation

$$R = \sigma T^4 = \left(5.6705 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4} \right) (2.7 \text{ K})^4 = 3.014 \times 10^{-6} \frac{\text{W}}{\text{m}^2}. \quad (2)$$

Multiply this by the area of Earth,

$$A_{\text{Earth}} = 4\pi R_{\text{Earth}}^2 = 4(3.1416)(6.37 \times 10^6 \text{ m})^2 = 5.10 \times 10^{14} \text{ m}^2, \quad (3)$$

to get the total power

$$P = RA = \left(3.014 \times 10^{-6} \frac{\text{W}}{\text{m}^2} \right) (5.10 \times 10^{14} \text{ m}^2) = 1.54 \times 10^9 \text{ W}. \quad (4)$$

Problem 2: Tipler 3-18

$$(a) \text{ Equation 3-23: } \bar{E} = \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} = \frac{hc/(10hc/kT)}{e^{(hc/kT)/(10hc/kT)} - 1} = \frac{0.1kT}{e^{0.1} - 1} = 0.951 kT$$

$$(b) \bar{E} = \frac{hc/\lambda}{e^{hc/\lambda kT} - 1} = \frac{hc/(0.1hc/kT)}{e^{(hc/kT)/(0.1hc/kT)} - 1} = \frac{10kT}{e^{10} - 1} = 4.59 \times 10^{-4} kT.$$

Equipartition theorem predicts $\bar{E} = kT$. The long wavelength value is very close to kT , but the short wavelength value is much smaller than the classical prediction.

Problem 3: Tipler 3-19

$$(a) \lambda_m T = 2.898 \times 10^{-3} m \cdot K \therefore T_1 = \frac{2.898 \times 10^{-3} m \cdot K}{27.0 \times 10^{-6} m} = 107 K$$

$$R_1 = \sigma T_1^4 \text{ and } R_2 = \sigma T_2^4 = 2R_1 = 2\sigma T_1^4$$

$$\therefore T_2^4 = 2T_1^4 \text{ or } T_2 = 2^{1/4} T_1 = (2^{1/4})(107 K) = 128 K$$

$$(b) \lambda_m = \frac{2.898 \times 10^{-3} m \cdot K}{128 K} = 23 \times 10^{-6} m$$

Problem 4: Tipler 3-21

Equation 3-10:

$$R = \sigma T^4$$

$$P_{abs} = (1.36 \times 10^3 W/m^2)(\pi R_E^2 m^2) \text{ where } R_E = \text{radius of Earth}$$

$$P_{emit} = (R W/m^2)(4\pi R_E^2) = (1.36 \times 10^3 W/m^2)(\pi R_E^2 m^2)$$

$$R = (1.36 \times 10^3 W/m^2) \left(\frac{\pi R_E^2}{4\pi R_E^2} \right) = \frac{1.36 \times 10^3 W}{4 m^2} = \sigma T^4$$

$$T^4 = \frac{1.36 \times 10^3 W/m^2}{4(5.67 \times 10^{-8} W/m^2 \cdot K^4)} \therefore T = 278.3 K = 5.3 C$$

Notice on the second line, one multiplies the given intensity (power/area) of the incident solar radiation by the *projected* area of Earth, πR_E^2 , the cross-sectional area that Earth presents toward the Sun. That gives the total power absorbed by the side of Earth facing the Sun. (The m^2 in that equation stands for meters²; it isn't part of the formula!) Then on the third line, when calculating the total radiation emitted by Earth, one multiplies by the *total* surface area of the spherical Earth $4\pi R_E^2$.