

2006-11-29 Problem Solutions

Problem 1: Tipler 6-41

(a) For $x > 0$, $\hbar^2 k_2^2 / 2m + V_0 = E = \hbar^2 k_1^2 / 2m = 2V_0$

So, $k_2 = (2mV_0)^{1/2} / \hbar$. Because $k_1 = (4mV_0)^{1/2} / \hbar$, then $k_2 = k_1 / \sqrt{2}$.

(b) $R = (k_1 - k_2)^2 / (k_1 + k_2)^2$ (Equation 6-68)

$= (1 - 1/\sqrt{2})^2 / (1 + 1/\sqrt{2})^2 = 0.0294$, or 2.94% of the incident particles are reflected.

(c) $T = 1 - R = 1 - 0.0294 = 0.971$

(d) 97.1% of the particles, or $0.971 \times 10^6 = 9.71 \times 10^5$, continue past the step in the $+x$ direction.

Classically, 100% would continue on.

Problem 2: Tipler 6-42

(a) For $x > 0$, $\hbar^2 k_2^2 / 2m - V_0 = E = \hbar^2 k_1^2 / 2m = 2V_0$

So, $k_2 = (6mV_0)^{1/2} / \hbar$. Because $k_1 = (4mV_0)^{1/2} / \hbar$, then $k_2 = \sqrt{3/2} k_1$.

(b) $R = (k_1 - k_2)^2 / (k_1 + k_2)^2 = (1 - \sqrt{3/2})^2 / (1 + \sqrt{3/2})^2 = 0.0102$. Or 1.02% are reflected at $x = 0$.

(c) $T = 1 - R = 0.99$

(d) 99% of the particles, or $0.99 \times 10^6 = 9.9 \times 10^5$, continue in the $+x$ direction. Classically, 100% would continue on.

Problem 3: Tipler 6-46

Using Equation 6-76, $T \approx 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0} \right) e^{-2\alpha a}$ where $E = 2.0 \text{ eV}$, $V_0 = 6.5 \text{ eV}$, and

$$a = 0.5 \text{ nm. } T \approx 16 \left(\frac{2.0}{6.5} \right) \left(1 - \frac{2.0}{6.5} \right) e^{-2(10.87)(0.5)} \approx 6.5 \times 10^{-5}$$

(Equation 6-75 yields $T = 6.6 \times 10^{-5}$.)

Problem 4:

a) The wave function in the three regions is

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x}, & x < 0, \\ Ce^{ik_2x} + De^{-ik_2x}, & 0 < x < a, \\ Fe^{ik_1x}, & a < x. \end{cases} \quad (1)$$

The definitions of the wavevectors k_1 and k_2 are

$$k_1 = \sqrt{\frac{2m}{\hbar^2} E}, \quad k_2 = \sqrt{\frac{2m}{\hbar^2} [E - V_0]}. \quad (2)$$

The wave function in Eq. (1) has propagating waves in all three regions. Equation (1) also incorporates the boundary condition that there is only a right-going wave on the right.

b) For this case of $E > V_0$, the transmission coefficient is

$$T = \frac{1}{1 + \frac{\sin^2(k_2 a)}{\frac{E}{V_0} \left(1 - \frac{E}{V_0} \right)}}, \quad (3)$$

i.e. the hyperbolic sine is replaced by the sine and α is replaced by k_2 (because that's how the wave function changed in the interval with the barrier; see Eq. (1)).

In order for T to be equal to 1, the 2nd term in the denominator of Eq. (3) must be zero. The only way for that term to be zero is for $\sin(k_2 a) = 0$. The sine function is zero when its argument is an integer multiple of π ,

$$k_2 a = n\pi, \quad n = 1, 2, 3, \dots \quad (4)$$

The wave vector is related to the wavelength (in region II) by $k_2 = 2\pi / \lambda_2$, so

$$\frac{2\pi}{\lambda_2} a = n\pi \quad (5)$$

or

$$\lambda_2 = \frac{2a}{n}, \quad n = 1, 2, 3, \dots \quad (6)$$

Rewrite Eq. (6) as

$$n \frac{\lambda_2}{2} = a. \quad (7)$$

This equation says that an integer number of half-wavelengths fit in the interval with the barrier, i.e. the wave “fits” into that interval. This is a resonance condition; the waves reflect back and forth from the edges of the barrier and undergo constructive interference. That makes the transmission coefficient as large as it possibly can be.